Generalized Difference Sequence Spaces Defined By $|\bar{N}, p_k|$ Summability and Orlicz Functions in Seminormed Space

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Abstract—*The object of this paper is to introduce a new difference sequence spaces which arise from the notions of* $|\bar{N}, p_k|$ *summability, using an infinite matrix* B *and an orlicz function in seminormed complex linear space. Various algebraic and topological properties and certain inclusion relations involving this space have been discussed.*

Keywords*: Difference sequence space, orlicz function, infinite matrix, topological linear space.*

1. INTRODUCTION

Let (s_k) denotes the sequence of partial sums of the infinite series $\sum_{k=0}^{\infty} a_k$. Denote by $(p_k), k \geq 0$ a sequence of positive real numbers and write $P_k = \sum_{z=o}^{k} p_z$. The series $\sum_{k=0}^{\infty} a_k$ (or the sequence (s_k)) is said to be summable $(\bar{N}, p_k)$ to the sum ℓ (finite), if

$$t_k = \frac{1}{P_k} \sum_{z=o}^{k} p_z s_z \to \ell \quad as \quad k \to \infty,$$

and is said to be absolutely summable $(\bar{N}, p_k)$ or summable $|\bar{N}, p_k|$ if the sequence $(t_k) \in BV$, that is $\sum_k |t_k - t_{k-1}| < \infty$. Let $|\bar{N}_p|$ and $\bar{N}_p$ denote respectively, the set of all sequences which are summable $|\bar{N}, p_k|$ and $(\bar{N}, p_k)$.

Given a sequence $a = (a_k)$, write for $k \geq 1$, $\phi_k(a) = t_k - t_{k-1}$. By an application of Abel's transformation we have

$$\varphi_k(a) = t_k - t_{k-1} = \frac{p_k}{P_k P_{k-1}} \sum_{n=1}^{k} P_{n-1} a_n \quad (k \geq 1).$$

Note that for any sequences a, b and scalar λ, we have

$$\phi_k(a+b) = \varphi_k(a) + \varphi_k(b) \quad \text{and} \quad \phi_k(\lambda a) = \lambda \varphi_k(a).$$

Lindenstrauss and Tzafriri [10] used the idea of an orlicz function M to construct the sequence space ℓ_M of all sequences of scalars (x_k) such that

$$\sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \quad \text{for some } \rho > 0.$$

The space ℓ_M equipped with the norm

$$\|x\| = \inf\left\{ \rho > 0 \ : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

is a BK space [7, p. 300] usually called an orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p which is an orlicz sequence space with $M(x)=x^p$, for $1 \le p < \infty$. We recall [7,10] that an orlicz function M is a function $[0,\infty) \to [0,\infty)$, which is continuous, non-decreasing and convex with $M(0)=0$, $M(x)>0$ for all $x>0$ and $M(x)\to\infty$, as $x\to\infty$. Note that an orlicz function is always unbounded .

An orlicz function M is said to satisfy $\Delta_2 - condition$ for all values of u, if there exists constant K such that $M(2u) \le K M(u), u \ge 0$. It is easy to see that always $K > 2$ [8]. A simple example of an orlicz function which satisfies the $\Delta_2 - condition$ for all values of u is given by $M(u) = a|u|^{\alpha}(\alpha > 1)$, since $M(2u) = a2^{\alpha}|u|^{\alpha} = 2^{\alpha} M(u)$. The orlicz function $M(u) = e^{|u|} - |u| - 1$ does not satisfy the $\Delta_2 - condition$.

The $\Delta_2 - condition$ is equivalent to the inequality $M(\ell u) \le K(\ell)M(u)$ which holds for all values of u, where ℓ can be any number greater than unity. It is easy to see that $M_1 + M_2$ is an orlicz function when M_1 and M_2 are orlicz functions, and that the function M^z (z is a positive integer), the composition of an orlicz function M with itself z times, is also an orlicz function. If an orlicz function M satisfies the $\Delta_2 - condition$, then so does the composite orlicz function M^z

By w we shall denote the space of all scalar sequences. ℓ_∞, c *and* c_0 denote the spaces of bounded, convergent and null sequences $x=(x_k)$ with complex terms, respectively, normed by $\|x\| = \sup_k |x_k|$. The notion of difference sequence spaces was introduced by kizmaz [8]. It was further generalized by Et and Colak [4] . Later on Et and Esi [5] defined the sequence spaces

$$X(\Delta_v^m) = \{ x = (x_k) \in w : (\Delta_v^m x_k) \in X \}$$

where $m \in N$, $\Delta_v^0 x_k = v_k x_k$, $\Delta_v x = v_k x_k - v_{k+1} x_{k+1}$, $\Delta_v^m x = (\Delta_v^m x_k) = (\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1})$ so that

$$\Delta_v^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+i} x_{k+i} .$$

Throughout the paper X denotes a seminormed complex linear space with seminorm q, M is an orlicz function, $s \ge 0$ is a real number, $r = (r_k)$ is a bounded sequence of strictly positive real numbers and $B = (b_{nk})$ of infinite matrix. The symbol $w(X)$ denotes the space of all X-valued sequences.

We now introduce the following generalized difference X-valued sequence space using orlicz function M .

$$|\bar{N}_p|(B, \Delta_v^m, M, r, q, s) =$$

$$\left\{ \begin{array}{l} a \in w(X) : \sum_{k=1}^{\infty} \frac{b_{nk}}{k^s} \left[M\left(q\left(\frac{\Delta_v^m \phi_k(a)}{\rho} \right) \right) \right]^{r_k} < \infty, \\ \text{for } some\ \rho > 0 \end{array} \right\}.$$

where $\Delta_v^m \varphi_k(a) = \Delta_v^{m-1} \varphi_k(a) - \Delta_v^{m-1} \varphi_{k+1}(a)$.

Some well-known spaces are obtained by specializing B, X, M, m, v, q, r, p *and* s.

(i) If $B = (C,1)$, that is the Cesaro matrix, $X = C$, $q(x) = |x|$, $m = v = 0$, $M(x) = x$ *and* $s = 0$ then $|\bar{N}_p|(\Delta_v^m, M, r, q, s) = |\bar{N}_p|(r)$

(Bhardwaj and Singh [2]).

(ii) If $B = (C,1)$, that is the Cesaro matrix, $X = C$, $q(x) = |x|$, $m = v = 0$ *and* $s = 0$ then $|\bar{N}_p|(\Delta_v^m, M, r, q, s) = |\bar{N}_p|(M, r)$

(Bhardwaj and Singh [3]).

(iii) If $B=(C,1)$, that is the Cesaro matrix, $m=v=0$, then $|\bar{N}_p|(\Delta_v^m, M, r, q, s)=|\bar{N}_p|(M, r, q, s)$ (Altin et al. [1]).

(iv) If $B=(C,1)$, that is the Cesaro matrix, $X=C, q(x)=|x|, m=v=0, M(x)=x, s=0$ *and* $p_k=1$ *for all* k, then

$|\bar{N}_p|(\Delta_v^m, M, r, q, s)=|C_1|(r)$. (Nanda and Mohanty [12]).

(v) If $B=(C,1)$, that is the Cesaro matrix, $X=C$, $q(x)=|x|, m=v=0, M(x)=x, s=0$ *and* $r_k=1$ *for all* k, then $|\bar{N}_p|(B, \Delta_v^m, M, r, q, s)=|\bar{N}_p|$.

We denote $|\bar{N}_p|(B, \Delta_v^m, M, r, q, s)$ by $|\bar{N}_p|(B, \Delta_v^m, r, q, s)$ when $M(x)=x$ and by $|\bar{N}_p|(B, \Delta_v^m, M, r, q)$ when $s=0$.

In this paper, we propose to study the linear topological structure of the sequence space

$|\bar{N}_p|(B, \Delta_v^m, M, r, q, s)$, certain inclusion relations between these spaces have been discussed. The composite space $|\bar{N}_p|(B, \Delta_v^m, M^z, r, q, s)$ using composite orlicz function M^z has also been studied.

The following inequalities [11, p. 190] are needed throughout the paper.

Let $r=(r_k)$ be a bounded sequence of strictly positive real numbers. If $H=\sup_k r_k$ then for any complex a_k, b_k,

$$|a_k+b_k|^{r_k} \le D\left(|a_k|^{r_k}+|b_k|^{r_k}\right) \quad (1)$$

where $D=\max\{1, 2^{H-1}\}$. Also for any complex λ,

$$|\lambda|^{r_k} \le \max\{1, |\lambda|^H\} \quad (2)$$

2. LINEAR TOPOLOGICAL STRUCTURE OF $|\bar{N}_p|(B, \Delta_v^m, M, r, q, s)$ SPACE AND INCLUSION THEOREMS

In this section we examine various algebraic and topological properties of the space $|\bar{N}_p|(B, \Delta_v^m, M, r, q, s)$ and investigate some inclusion relations.

Theorem 2.1. For any Orlicz function M, $|\bar{N}_p|(B, \Delta_v^m, M, r, q, s)$ is a linear space over the complex field C.

The proof is a routine verification by using standard techniques and hence is omitted.

Theorem 2.2. For any Orlicz function M, $|\bar{N}_p|(B, \Delta_v^m, M, r, q, s)$ is a is a topological linear space, paranormed by

$$g_\Delta(a)=\inf\left\{\rho^{r_n/G} : \left(\sum_{k=1}^{\infty}\frac{b_{nk}}{k^s}\left[M\left(q\left(\frac{\Delta_v^m\phi_k(a)}{\rho}\right)\right)\right]^{r_k}\right)^{1/G} \le 1, \; n=1,2,\ldots\right\}.$$

where $G=\max\left(1, \sup_k r_k\right)$.

The proof uses ideas similar to those used (e.g.) in [1, p. 427] and the fact that every paranormed space is a topological linear space [13, p. 37].

Remark 2.3. g_Δ need not be total, for example if $p_k=1$ for all k and $a=a_k$ is any non-zero constant sequence then $\phi_k(a)$ is constant for all k and hence $g_\Delta(a)$ is zero for $m \ge 1$.

Theorem 2.4. Let M, M_1, M_2 be orlicz functions, then

(i) If there is a positive constant β such that $M(t) \le \beta t$ for all $t \ge 0$, then

$$|\bar{N}_p|(B, \Delta_v^m, M, r, q, s) \subseteq |\bar{N}_p|(B, \Delta_v^m, M \circ M_1, r, q, s).$$

$$(ii)\,|\bar{N}_p|\left(B,\Delta_v^m,M_1,r,q,s\right)$$
$$\bigcap|\bar{N}_p|\left(B,\Delta_v^m,M_2,r,q,s\right)$$
$$\subseteq|\bar{N}_p|\left(B,\Delta_v^m,M_1+M_2,r,q,s\right).$$

Proof: (i) Let $a\in|\bar{N}_p|\left(B,\Delta_v^m,M_1,r,q,s\right)$ so that $\sum_{k=1}^{\infty}\frac{b_{nk}}{k^s}\left[M_1\left(q\left(\frac{\Delta_v^m\phi_k(a)}{\rho}\right)\right)\right]^{r_k}<\infty$ for some $\rho>0$. Since $M(t)\le\beta t$ for all $t\ge 0$, we have by inequality (2)

$$\sum_{k=1}^{\infty}\frac{b_{nk}}{k^s}\left[Mu_k\right]^{r_k}\le\max\left(1,\beta^H\right)\sum_{k=1}^{\infty}\frac{b_{nk}}{k^s}\left[u_k\right]^{r_k}$$

where $u_k=M_1\left(q\left(\frac{\Delta_v^m\phi_k(a)}{\rho}\right)\right)$ and hence $a\in|\bar{N}_p|\left(B,\Delta_v^m,M\circ M_1,r,q,s\right)$.

(ii) The proof is immediate using (1).

Theorem 2.5. For any orlicz function M if $\lim_{u\to\infty}\frac{M(u/\rho)}{(u/\rho)}>0$ for some $\rho>0$, then

$$|\bar{N}_p|\left(B,\Delta_v^m,M,r,q,s\right)\subseteq|\bar{N}_p|\left(B,\Delta_v^m,r,q,s\right).$$

Proof: If $\lim_{u\to\infty}\frac{M(u/\rho)}{(u/\rho)}>0$ for some $\rho>0$, there exists a number $\alpha>0$ such $M(u/\rho)\ge\alpha\,(u/\rho)$ for all $u>0$ and some $\rho>0$. Let $a\in|\bar{N}_p|\left(B,\Delta_v^m,M,r,q,s\right)$ so that $\sum_{k=1}^{\infty}\frac{b_{nk}}{k^s}\left[M\left(q\left(\frac{\Delta_v^m\phi_k(a)}{\rho}\right)\right)\right]^{r_k}<\infty$ for some $\rho>0$.

Now , we have

$$\sum_{k=1}^{\infty}\frac{b_{nk}}{k^s}\left[M\left(q\left(\frac{\Delta_v^m\varphi_k(a)}{\rho}\right)\right)\right]^{r_k}\ge$$
$$\max\left(1,\left(\frac{\alpha}{\rho}\right)^H\right)\sum_{k=1}^{\infty}\frac{b_{nk}}{k^s}\left[q\left(\Delta_v^m\varphi_k(a)\right)\right]^{r_k}.$$

Hence $a\in|\bar{N}_p|\left(B,\Delta_v^m,r,q,s\right)$.

Theorem 2.6. Let M be an orlicz function which satisfies $\Delta_2-condition$, for some q,q_1,q_2 be seminorms and s,s_1,s_2 be non-negative real numbers . Then

$$(i)\,|\bar{N}_p|\left(B,\Delta_v^m,M,r,q_1,s\right)$$
$$\bigcap|\bar{N}_p|\left(B,\Delta_v^m,M,r,q_2,s\right)$$
$$\subseteq|\bar{N}_p|\left(B,\Delta_v^m,M,r,q_1+q_2,s\right).$$

(ii) If there exists a constant $L>1$ such that $q_2(x)\le L\,q_1(x)$ for all $x\in X$, then

$$|\bar{N}_p|\left(B,\Delta_v^m,M,r,q_1,s\right)$$
$$\subseteq|\bar{N}_p|\left(B,\Delta_v^m,M,r,q_2,s\right).$$

(iii) If $s_1\le s_2$, then

$$|\bar{N}_p|\left(B,\Delta_v^m,M,r,q,s_1\right)\subseteq|\bar{N}_p|\left(B,\Delta_v^m,M,r,q,s_2\right).$$

Proof: The proof of (i) is straight forward using (1).

(ii) Let $a\in|\bar{N}_p|\left(B,\Delta_v^m,M,r,q_1,s\right)$. Then as M satisfies $\Delta_2-condition$, we have

$$\sum_{k=1}^{\infty}\frac{b_{nk}}{k^s}\left[M\left(q_2\left(\frac{\Delta_v^m\phi_k(a)}{\rho}\right)\right)\right]^{r_k}\le$$
$$\sum_{k=1}^{\infty}\frac{b_{nk}}{k^s}\left[M\left(L\,q_1\left(\frac{\Delta_v^m\phi_k(a)}{\rho}\right)\right)\right]^{r_k}$$

$$\leq \sum_{k=1}^{\infty} \frac{b_{nk}}{k^s}\left[K(L)M\left(q_1\left(\frac{\Delta_v^m \varphi_k(a)}{\rho}\right)\right)\right]^{r_k}$$

$$\leq \max\left(1, K(L)^H\right)\sum_{k=1}^{\infty}\frac{b_{nk}}{k^s}\left[M\left(q_1\left(\frac{\Delta_v^m \varphi_k(a)}{\rho}\right)\right)\right]^{r_k} < \infty.$$

Hence $a \in |\bar{N}_p|\left(B, \Delta_v^m, r, q_2, s\right)$.

The proof of (iii) is trivial.

Theorem 2.7. Let $m \geq 1$, then

$$|\bar{N}_p|\left(B, \Delta_v^{m-1}, M, q, s\right) \subseteq |\bar{N}_p|\left(B, \Delta_v^m, M, q, s\right).$$

Proof: Let $a \in |\bar{N}_p|\left(B, \Delta_v^{m-1}, M, q, s\right)$, then $\sum_{k=1}^{\infty}\frac{b_{nk}}{k^s}\left[M\left(q\left(\frac{\Delta_v^{m-1}\phi_k(a)}{\rho}\right)\right)\right] < \infty$ for some $\rho > 0$. Let $\rho_1 = 2\rho$, then as q is seminorm and M is non-decreasing and convex, we have

$$\sum_{k=1}^{\infty}\frac{b_{nk}}{k^s}\left[M\left(q\left(\frac{\Delta_v^m \phi_k(a)}{\rho}\right)\right)\right]$$

$$\leq \sum_{k=1}^{\infty}\frac{b_{nk}}{k^s}\left[\frac{M}{2}\left(q\left(\frac{\Delta_v^{m-1}\varphi_k(a)}{\rho}\right)\right) + \frac{1}{2}M\left(q\left(\frac{\Delta_v^{m-1}\varphi_{k+1}(a)}{\rho}\right)\right)\right]$$

whence $a \in |\bar{N}_p|\left(B, \Delta_v^m, M, q, s\right)$.

In general $|\bar{N}_p|\left(B, \Delta_v^i, M, q, s\right) \subseteq |\bar{N}_p|\left(B, \Delta_v^m, M, q, s\right)$ for all $i = 1, 2, \ldots\ m-1$ and the inclusion is strict. To show that the inclusion is strict, consider the following example.

Example 2.8. Let $X = C, q(x) = |x|, M(x) = x$, $s = 0$ and $p_k = 1$ for all k. Let $a = (a_k)$ be defined by $a_k = 3k^2 - 3k + 1$, then $a \notin |\bar{N}_p|\left(B, \Delta_v^2, M, q, s\right)$ but $a \in |\bar{N}_p|\left(B, \Delta_v^3, M, q, s\right)$.

Theorem 2.9. If $t = (t_k)$ *and* $r = (r_k)$ are bounded sequences of positive real numbers with $0 < t_k \leq r_k < \infty$ for each k, then for any orlicz function M,

$$|\bar{N}_p|\left(B, \Delta_v^m, M, t, q\right) \subseteq |\bar{N}_p|\left(B, \Delta_v^m, M, r, q\right).$$

Proof: Let $a \in |\bar{N}_p|\left(B, \Delta_v^m, M, t, q\right)$. Then there exists some $\rho > 0$ such that $\sum_{k=1}^{\infty} b_{nk}\left[M\left(q\left(\frac{\Delta_v^m \phi_k(a)}{\rho}\right)\right)\right]^{t_k} < \infty$. This implies that $b_{nk} M\left(q\left(\frac{\Delta_v^m \phi_i(a)}{\rho}\right)\right) \leq 1$ for sufficiently large values of i say $i \geq k_0$ for some fixed $k_0 \in N$. Since M is non-decreasing, we get

$$\sum_{k \geq k_0}^{\infty} b_{nk}\left[M\left(q\left(\frac{\Delta_v^m \phi_k(a)}{\rho}\right)\right)\right]^{r_k} \leq \sum_{k \geq k_0}^{\infty} b_{nk}\left[M\left(q\left(\frac{\Delta_v^m \phi_k(a)}{\rho}\right)\right)\right]^{t_k} < \infty.$$

Hence $a \in |\bar{N}_p|\left(B, \Delta_v^m, M, r, q\right)$.

3. Composite space $|\bar{N}_p|\left(B, \Delta_v^m, M^z, r, q, s\right)$ using composite orlicz function M^z

Taking orlicz function M^z instead of M in the space $|\bar{N}_p|\left(B, \Delta_v^m, M, r, q, s\right)$ we can define the composite space $|\bar{N}_p|\left(B, \Delta_v^m, M^z, r, q, s\right)$ as follows:

Definition 3.1. For a fixed natural number z, we define

$$|\bar{N}_p|\left(B,\Delta_v^m,M^z,r,q,s\right)=$$

$$\left\{\begin{array}{l} a\in w(X):\sum_{k=1}^{\infty}\frac{b_{nk}}{k^s}\left[M^z\left(q\left(\frac{\Delta_v^m\phi_k(a)}{\rho}\right)\right)\right]^{r_k} \\ <\infty,\ for\ some\ \rho>0 \end{array}\right\}.$$

Theorem 3.2. For any orlicz function M and $z\in N$,

$(i)|\bar{N}_p|\left(B,\Delta_v^m,M^z,r,q,s\right)$

$\subseteq|\bar{N}_p|\left(B,\ \Delta_v^m,\ r,q,s\right).$

(ii) Suppose there exists a constant β , $0<\beta\le 1$, such that $M(t)\le\beta(t)$ for all $t\ge 0$ and let and $n,z\in N$ be such that $n<z$, then

$$|\bar{N}_p|\left(B,\ \Delta_v^m,\ r,q,s\right)\subseteq|\bar{N}_p|\left(B,\ \Delta_v^m,M^n,r,q,s\right)$$

$$\subseteq|\bar{N}_p|\left(B,\ \Delta_v^m,M^z,r,q,s\right).$$

The proof follows from Theorem 2.5 and Theorem 2.4(i) and hence is omitted.

Example 3.3. $M_1(t)=e^t-1\ge 1$ and $M_2(t)=\frac{t^2}{1+t}\le t$ for all $t\ge 0$ satisfy the conditions given in Theorem 3.2 (i), (ii) respectively.

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